A Word on the Frenet–Serret Frame $\{\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t)\}$

Let's say we want to find the unit tangent, unit normal, and binormal vectors of a curve $\mathbf{r}(t)$ in $\mathbb{R}^3$. Of course the unit tangent vector is simply

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|},$$

and this is really the only practical way to compute this. On the other hand, when computing the unit normal and the binormal vectors, we have a bit of flexability in which order we compute them. Following the method of the book, we find that

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

and

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t).$$

However, this can be rather cumbersome to carry out, so let's look at some other ways to compute these things. If you compute just $\mathbf{T}'(t)$, we get

$$\mathbf{T}'(t) = \frac{\mathbf{r}''(t) - (\mathbf{r}'(t) \cdot \mathbf{r}''(t))\,\mathbf{r}'(t)}{|\mathbf{r}'(t)|^3}$$

which is a vector pointing in the same direction as $\mathbf{N}(t)$, just maybe not with the same length. Whatever the length of $\mathbf{T}'(t)$ may be, thinking geometrically about this, we know that $\mathbf{T}(t) \times \mathbf{T}'(t)$ points in the same direction as $\mathbf{B}(t)$ (again, it probably has a different length). Now, if you'll observe the cross product $\mathbf{T}(t) \times \mathbf{T}'(t)$ carefully, you'll see that

$$\mathbf{T}(t) \times \mathbf{T}'(t) = \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{|\mathbf{r}'(t)|^4}$$

by the properties $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$ and $c\mathbf{a} \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b})$, and because $\mathbf{r}'(t) \times \mathbf{r}'(t) = \mathbf{0}$. Thus $\mathbf{B}(t)$ points in the same direction as $\mathbf{r}'(t) \times \mathbf{r}''(t)$. Now, we know that both $\mathbf{T}(t)$ and $\mathbf{N}(t)$ are unit vectors and that they are orthogonal to each other, i.e., the angle between them is $\frac{\pi}{2}$. Using the fact that $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$, we can see that

$$|\mathbf{B}(t)| = |\mathbf{T}(t)|\,|\mathbf{N}(t)| \sin\frac{\pi}{2} = 1$$

so that $\mathbf{B}(t)$ is also a unit vector. This is good news! Since we already have a vector pointing in the direction of $\mathbf{B}(t)$, all we need to do now to get $\mathbf{B}(t)$ is normalize that vector. Thus we have

$$\mathbf{B}(t) = \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}.$$

Now that we have $\mathbf{B}(t)$, what remains to be computed is $\mathbf{N}(t)$. Recall that the cross product has a "cyclic property" for orthogonal unit vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, that is

$$\mathbf{a} = \mathbf{b} \times \mathbf{c} \Longrightarrow \mathbf{b} = \mathbf{c} \times \mathbf{a} \Longrightarrow \mathbf{c} = \mathbf{a} \times \mathbf{b} \Longrightarrow \mathbf{a} = \mathbf{b} \times \mathbf{c}$$

so since $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$, we get that

$$\mathbf{N}(t) = \mathbf{B}(t) \times \mathbf{T}(t).$$

In summary, we may use either of the following two methods to find the Frenet-Serret frame of a curve $\mathbf{r}(t)$:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \qquad \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \qquad \mathbf{B}(t) = \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}$$

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) \qquad \mathbf{N}(t) = \mathbf{B}(t) \times \mathbf{T}(t)$$